Interference-Constrained Wireless Coverage in a Protocol Model

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ABSTRACT

We present an efficient algorithm to compute the coverage map of a given set of transmitters under interference constraints. That is, we compute the set of points that lie within the transmission range of one transmitter and lie outside the interference range of every other transmitter. To our knowledge, there is no existing satisfactory algorithm for this purpose. We assume that the transmission and interference ranges of each transmitter are circular disks centered at the transmitter location.

We show that for an appropriate choice of 'distance measure', coverage at each point can be computed by considering only certain 'proximate' transmitters. Hence, we partition the plane into proximity regions and the coverage in these proximity regions is computed considering only proximate transmitters. Our algorithm takes $O(n \log n)$ time. We use Voronoi diagrams and power diagrams for representing these proximity regions.

1. INTRODUCTION

The wireless network designer is often faced with the task of determining the coverage regions of a set of transmitters under interference. The transmitters may correspond to base stations in an infrastructure network, wireless access points in a wireless local area network, or transmitter nodes in a wireless ad-hoc network. The inputs to a designer are the transmitter locations, their transmission power, and the channel model (modulation scheme, path loss, fading, etc.).

Co-channel interference occurs at a receiver during simultaneous wireless communication between two transmitter-receiver pairs. Each transmitter has a limited transmission range within which the intended receiver must lie. Each transmitter also has an interference range - a range typically larger and inclusive of the transmission range. A receiver that lies within the transmission range of transmitter A and the interference range of transmitter B cannot receive (or, more precisely, decode received signals) from transmitter A if both transmitters A and B transmit simultaneously.

Suppose the coverage map of a given set of transmitters is known. The designer can attempt to improve coverage by varying the location of the transmitters, or their transmission power. Recently, Ahmed et al. ([4]) have proposed algorithms for transmission power assignment to access points (APs) under interference constraints. Their solutions also assume a protocol model similar to ours. They describe the computation of exact coverage to be a 'mathematically daunting' task. Currently, to the best of our knowledge, there is no satisfactory algorithm that finds the coverage map of a wireless network under interference constraints. Our purpose in this work is to compute the coverage map efficiently using simple computational geometric primitives.

1.1 Problem Statement

We are given n transmitter locations (points) in the plane. We are also given the transmission and interference ranges of each transmitter. All transmitters share the same wireless channel. We need to compute the set of points that lie within the transmission range of one transmitter, and outside the interference range of every other. We call this set the 'coverage region' of a transmitter. The union of all n coverage regions is the 'coverage map' of the network.

The rest of the paper is organized as follows: The remainder of this section describes our model and our solution approach. Sections 2 and 3 describe our solution for equal and unequal transmission ranges (respectively), Section 4 describes an algorithm for computing the coverage map and its analysis, and Section 5 discusses related works and di-

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rections for future work.

1.2 Model

Our solution makes the following assumptions in order to pose the problem in terms of simple extensions to well-known computational geometric ideas.

1. The transmission and interference ranges of each transmitter is a circular disk centered at the transmitter location. Also, the transmission range is less than the interference range. (We use the terms “range” and “disk” interchangeably in the subsequent text.)
2. Physical layer characteristics of the medium are not modeled. The model used is a “Protocol” Model, in the style of Gupta et al. ([9]).
3. Each receiver has the same receive sensitivity. This means that the interference and transmission ranges are independent of the physical characteristics of the receiver.

This model idealizes the omni-directional transmitter in the plane. We treat disks as open sets; i.e. technically, a point on the rim of the disk is said to lie outside the disk, whereas all points inside the rim lie on the disk.

In a protocol model, the coverage of a point is decided by set-membership alone - the sets being points inside appropriate transmission and interference ranges. We state our protocol model more formally :

1. Each transmitter $i \in \{1, \ldots, n\}$ has a corresponding location l_i in the 2-D plane, a transmission radius a_i, and an interference radius e_i.
2. The transmission and interference ranges, s_i and f_i respectively, are represented by the sets of points in the circles centered at l_i, with radii a_i and e_i respectively.
3. The coverage region, c_i, of transmitter i is the set

$$c_i = s_i \setminus \bigcup_{\substack{j \neq i \\ j \in \{1,\ldots,n\}}} f_j$$

.

Although the model as stated alludes to all points in the plane, we will restrict our attention to some sufficiently large subset of the plane without explicit mention.

The shaded region in figure 1 shows the coverage region of one transmitter surrounded by 7 other transmitters. The interference ranges (disks) are shown in solid perimeter and the transmission disk of transmitter p is shown with a dotted perimeter. Note that the transmission disk of other transmitters is a subset of the interference disk, and hence is not shown in the figure.

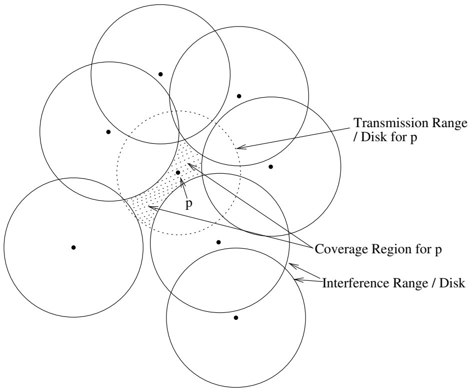

Figure 1: Coverage Region of a Transmitter

1.3 Solution Approach

We show that for an appropriate choice of ‘distance measure’, coverage at each point can be computed by considering only certain ‘proximate’ transmitters. Hence, we partition the plane into ‘proximity’ regions and the coverage in these proximity regions is computed considering only proximate transmitters. Our algorithm takes $O(n \log n)$ time.

We argue, in Sections 2 and 3 that the coverage region for a transmitter must necessarily lie in its proximity region. We further show that there exists a partition of this proximity region such that coverage in each partition can be decided by considering the interference from only one ‘nearby’ neighbor.

For ease of illustration, we first present our approach in Section 2 assuming that the transmission radius of all transmitters is the same; and also that the interference radius of all transmitters is the same. Section 3 relaxes this assumption to allow unequal transmission radii and interference radii. Our research also developed along these lines.

For equal ranges, we use the distance measure between a point and a disk as the Euclidean distance between the point and the center of the disk. The partition into proximity regions under this measure is well-known - it is called the *closest point Voronoi Diagram* ([2]). Our method is an augmentation of the Voronoi diagram.

For unequal ranges, we use a related structure called the *Power Diagram*. The power diagram is very similar in shape, structure, representation, and properties to the Voronoi diagram ([3]). We show in Section 3 how this allows us to carry over our arguments for equal ranges even to unequal ranges.

The Voronoi and power diagrams have been applied to wireless coverage problems in the context of sensor networks - by So et al. ([1]), and before them by Meguerdichian et al. ([10]). The work in [1] has been the main motivation and starting point for us. We discuss this work in some detail in Section 5. However, we are not aware of any published application of Voronoi or power diagrams to coverage under interference constraints.

2. EQUAL RANGES AND THE VORONOI DIAGRAM

We propose an augmentation to the Voronoi diagram of the point set corresponding to transmitter locations. This augmentation yields an efficient algorithm for computing the coverage map.

The Voronoi diagram (see, for example figure 2) of a given point set in 2-D partitions the plane into Voronoi regions. Each point in the point set corresponds to exactly one region. The defining characteristic of a Voronoi region corresponding to point p is that every point in this region is closer, in Euclidean distance, to p than to any other point in the given point set.

Aurenhammer et al.([2]) give a detailed treatment of Voronoi diagrams, including its properties and algorithms. In this paper we consider Voronoi regions to be open sets - we call Voronoi edges and vertices the 'extreme points' of these regions. Each Voronoi region can be represented by an ordered list of edges enclosing it.

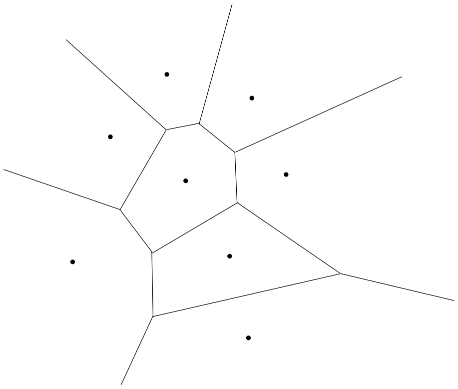

Figure 2: A Voronoi Diagram of 8 Points

Let T be the set of transmitter locations, and let $\mathbb{V}(T)$ denote their Voronoi diagram. Voronoi diagrams are classically used for reasoning about point sets. We show that we can apply Voronoi diagrams to our problem concerning interference disks with equal radii.

Let $\bigcirc(c, r)$ be a disk with center at c and radius r. We write $\bigcirc$ when the parameters c and r are clear from the context. Let δ be a distance measure between a point x and the disk $\bigcirc(c, r)$ - the minimum distance of x from the periphery of $\bigcirc$, wherein the distance is negative if $x \in \bigcirc$ and non-negative otherwise. Formally, $\delta(x, \bigcirc(c, r)) = d(p, c) - r$, where $d(x, c)$ denotes Euclidean distance between x and c.

OBSERVATION 2.1. *Let* $\bigcirc_1(c_1, r)$ *and* $\bigcirc_2(c_2, r)$ *be two disks of equal radii,* r*, with centers* c_1 *and* c_2 *respectively. For a given point* x*, let* δ_1 *and* δ_2 *be the minimum distances of* x *from the peripheries of* $\bigcirc_1$ *and* $\bigcirc_2$*, respectively.* x *is equidistant from the centers* c_1 *and* c_2 *if, and only if,* $\delta_1 = \delta_2$*.*

PROOF. Since $\delta_1 = \delta_2$, either both are positive, or both are negative. Let $x \in \bigcirc(c, r)$. Let y be a point on the periphery of $\bigcirc(c, r)$. By the triangle inequality $d(x, y) \geq r - d(x, c)$; i.e the minimum Euclidean distance possible is $r - d(x, c)$, which is when x, y, and c are collinear, with x between y and c.

Now,
$\delta_1 = \delta_2 \Leftrightarrow r - d(x, c_1) = r - d(x, c_2)$
$\Leftrightarrow d(x, c_1) = d(x, c_2)$

A similar observation can be made if p is outside both disks. $\square$

Observation 2.1 can also be extended to show that

$$\delta(x, \bigcirc(c, r)) > \delta(y, \bigcirc(c, r)) \Leftrightarrow d(x, c) > d(y, c)$$

Note that δ is a signed quantity.

We make an observation that relates the distance measure to interference disks.

OBSERVATION 2.2. *Consider a point* x *on the interference disk for* p*.* x *is also on the interference disk of every transmitter closer than* p*.*

PROOF. Let $q \in T \setminus \{p\}$. Let x be closer to q than p. Let $\bigcirc_p$ and $\bigcirc_q$ be the interference disks of p and q, respectively.

Since $d(x, q) < d(x, p) \Rightarrow \delta(x, \bigcirc_q) < \delta(x, \bigcirc_p)$, if x is in the interference range of p, $\delta(x, bigcirc_p) < 0$. Which means that $\delta(x, \bigcirc_q) < 0$, i.e. x must also be in the interference range of q. $\square$

The Voronoi diagram of the centers of disks tells us which center is closest to a given point, in Euclidean distance. Given disks with equal radii, let us have a diagram which partitions points in the plane according to the disk nearest to them by the distance measure δ; i.e. a 'Voronoi' diagram of disks. The above observation 2.1 shows that the 'Voronoi' diagram of disks of equal radii using the signed distance measure δ is the same as the Voronoi diagram of the centers of the respective disks. Thus we can compute the classical Voronoi diagram, using the Euclidean distance measure, instead of the signed minimum distances δ. The ensuing discussion in this section thus refers directly to the Voronoi diagram of the transmitter locations and Euclidean point distances from them.

We now state notation and expressions for the points corresponding to a Voronoi region, its extreme points, and the Voronoi diagram.

The Voronoi region corresponding to a point $p \in T$ is given by:

$$\triangle(p) = \{x \mid d(x, p) < d(x, q), \forall q \in T \setminus \{p\}\}$$

The extreme points of the Voronoi region for p are given by:

$$\partial(p) = \{x \mid d(x, p) \leq d(x, q), \forall q \in T \setminus \{p\}\} \setminus \triangle(p)$$

The Voronoi diagram is given by:

$$\mathbb{V}(T) = \bigcup_{p \in T} \partial(p)$$

We now prove an important property of the Voronoi region - it shows that the coverage region for a transmitter is confined to that transmitter's Voronoi region.

Lemma 2.1 (Coverage in Voronoi Region). *Consider a point x outside $\triangle(p)$ that is on the transmission disk for p. x is also on some interference disk other than that of p.*

Proof. Let $q \in T \setminus \{p\}$, and $x \in \triangle(q)$. The Voronoi property implies that p is farther away from x than q. The transmission disk of p is a subset of its interference disk; thus x lies on the interference disk of p. Hence, by observation 2.2, since x is in the interference range of p, x must also be in the interference range of q. ☐

Extreme points of $\triangle(p)$ appear as either Voronoi edges or vertices. These in turn correspond to some other transmitters in T that we call the set of Voronoi neighbors of p, denoted by $\Gamma(p)$. Lets assume that the Voronoi diagram for T has been built by some classical method ([2]). Suppose we delete the point p from T, and build $\mathbb{V}(T \setminus \{p\})$. Figure 3 shows the effect of these changes. :

1. Some existing edges are extended - see dotted portions in Figure 3
2. Some existing edges are deleted - see dashed edges in Figure 3
3. Some new vertices are added - see intersections of dotted extensions in Figure 3
4. Some new edges are added - see edges between new vertices in Figure 3

Note that in figure 3, only the neighborhood of p changes to form $\mathbb{V}(T \setminus \{p\})$. We will formally show later, in lemma 2.2, that this is true for every Voronoi diagram.

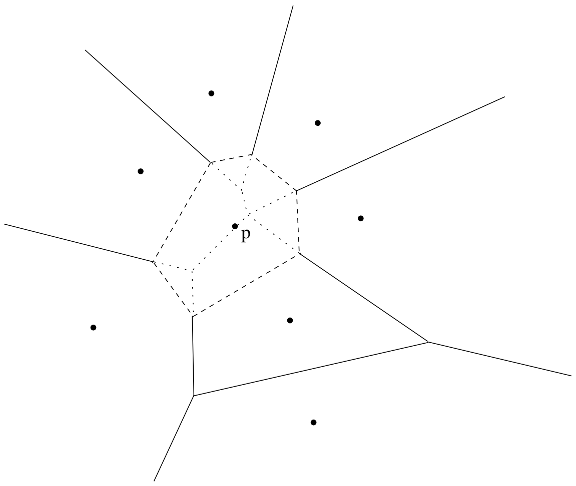

Figure 3: Effect of deleting p

We call the new set of vertices, edges, and extension edges the *Voronoi Frame* corresponding to p. The dotted portions in Figure 3 are the Voronoi Frame. Edges in the frame correspond to the set of points equidistant from two transmitters adjacent to p in $\mathbb{V}(T)$. Vertices in the frame correspond to points equidistant from three (or more) transmitters adjacent to p.

Before we delve into our discussion, we define certain terms in the context of what has been said so far.

Feasible Coverage Area: For a given transmitter p, this is the set of points lying in the Voronoi region of p, but outside every interference disk other than p. We reiterate that due to lemma 2.1, to compute coverage, we can restrict our attention only to the Voronoi region of p.

Actual Coverage Area: For a given transmitter p, this is the set of points that lie in its Feasible Coverage Area and also on its transmission disk. The Actual Coverage Area of p is the intersection of the transmission disk of p with the Feasible Coverage Area. An Actual Coverage Area is demarcated by arcs, each of which correspond to the portion of the periphery of either the transmission disk of p, or an interference disk of a Voronoi neighbor of p that bounds the Feasible Coverage Area.

Contiguous Feasible Region: The Feasible Coverage Area may be composed of several disjoint maximal simply-connected subsets (see, for example, the two shaded sets in figure 5). These subsets are called Contiguous Feasible Regions. In the subsequent text we use the term 'Feasible Region' when the word 'Contiguous' is obvious from the context.

Voronoi Frame: For a given transmitter p, this is the set of points on extension edges, new edges and vertices in $\triangle(p)$ obtained from deleting the point p and adding new extreme points from the Voronoi diagram of $T \setminus \{p\}$. The only extreme points in $\mathbb{V}(T)$ that do not belong in $\mathbb{V}(T \setminus \{p\})$ are the extreme points on the edges in $\triangle(p)$. The Voronoi Frame is thus the set of points in $\mathbb{V}(T \setminus \{p\}) \setminus \mathbb{V}(T)$.

Our goal in this remainder of this section is to demonstrate how the Voronoi Frame aids in reasoning about the Feasible and Actual Coverage Areas.

Figures 4, 5, and 6 will help the reader visualize the concepts being discussed. Each figure corresponds to the same set of transmitter locations. The interference radii, however, are different.

1. Interference disks are shown with solid perimeters.
2. Only one transmission disk is shown. It is shown by a dotted perimeter.
3. The original Voronoi diagram is shown by dashed lines.
4. The Voronoi Frame is shown by solid lines.

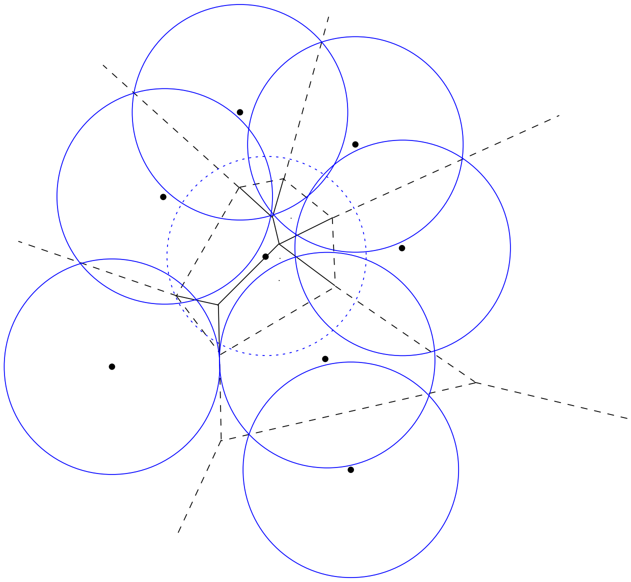

Figure 4: Feasible Coverage Area: Closed Feasible Region

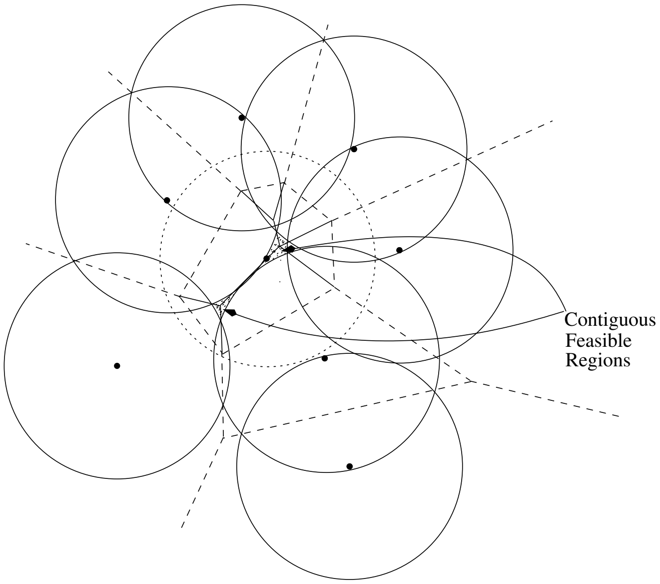

Figure 5: Feasible Coverage Area: Multiple Contiguous Feasible Regions

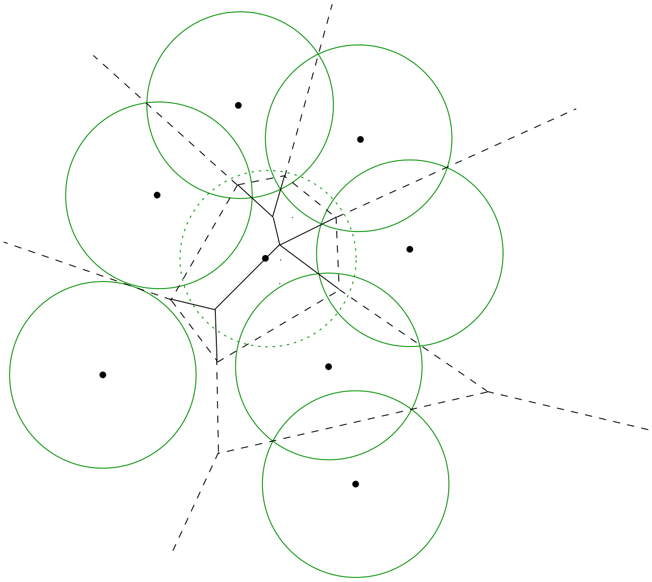

Figure 6: Feasible Coverage Area: Open Feasible Region

5. Figure 4 illustrates a closed Feasible Region; i.e. one that is bounded on all sides by interference disks.

6. Figure 5 illustrates two Contiguous Feasible Regions in one Feasible Coverage Area.

7. Figure 6 illustrates an open Feasible Region.

We now show that only neighbors of p in $\mathbb{V}(T)$ contribute edges in the Voronoi Frame for p. We actually observe a more general result - that all points in a Voronoi region are closer to Voronoi neighbors than to any other point -

LEMMA 2.2. *Let $x \in \triangle(p)$. The closest point to x in $T \setminus \{p\}$ is a Voronoi neighbor of p in $\mathbb{V}(T)$.*

PROOF. Let q be the closest point to x in $T \setminus \{p\}$. Assume that q is not a neighbor of p. We show that this leads to a contradiction. Note that p and q have distinct Voronoi regions, and each of them is a partition of the plane. Thus the line segment joining x and q must intersect an edge of $\partial(p)$. Let this intersection point be x_0, and the Voronoi neighbor of p on this edge be q_0. The position of the points is shown in figure 7.

Since p and q_0 are neighbors, $d(x_0, p) = d(x_0, q_0)$
$\Rightarrow$ {Since p and q are not neighbors, } $d(x_0, q) > d(x_0, p)$
$\Rightarrow$ {Adding $d(x, x_0)$ to both sides, } $d(x, x_0) + d(x_0, q) = d(x, q) > d(x, x_0) + d(x_0, q_0)$
$\Rightarrow$ {Since x, x_0, and q are colinear, } $d(x, q) > d(x, x_0) + d(x_0, q_0)$
$\Rightarrow$ {By the triangle inequality, } $d(x, x_0) + d(x_0, q_0) \geq d(x, q_0)$
$\Rightarrow d(x, q) > d(x, q_0)$
This contradicts the assumption that q is the closest point to x in $T \setminus \{p\}$. $\square$

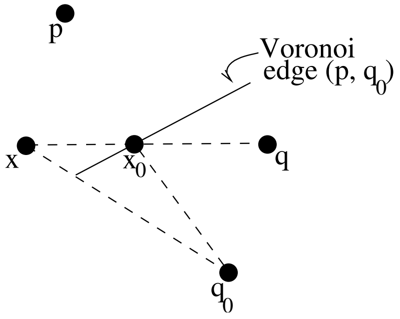

Figure 7: Illustration of Lemma 2.2

Corollary 2.2.1. *If a point $x \in \triangle(p)$ is in the interference range of some transmitter $t \neq p$, it is in the interference range of a neighbor of p.*

Proof. Lemma 2.2 states that the closest point to x in $T \setminus \{p\}$ is a neighbor of p. Thus, using observation 2.2 we infer that x is in the interference range of a neighbor of p. ☐

We denote by $\nu(x, S)$ the set of points in S closest to x.

We can now give an expression for the Voronoi Frame corresponding to p. Due to corollary 2.2.1, the Voronoi Frame has edges only from Voronoi neighbors :

$$\perp(p) = \{x \in \triangle(p) \mid \exists \{q_1, q_2\} \subseteq \nu(x, \Gamma(p))\}$$

Our aim is to compute the Actual Coverage Area for p. We will use the Voronoi Frame of p to do so. Note that not all points on the Voronoi Frame are in the Feasbile Coverage Area. This is because some interference disk corresponding to a neighbor may include part of an edge on the Voronoi Frame. Our first task is to exclude points on the Voronoi Frame that are on some interference disk in $T \setminus \{p\}$. We call the resultant subset of the Voronoi Frame the *Feasible Coverage Frame*. Corollary 2.2.1 shows that to obtain the Feasible Coverage Frame it is sufficient to exclude points on interference disks adjacent to edges in the Voronoi Frame. For each edge in the frame, we remove the portion of the edge on one of its adjacent interference disks. This results in zero, one, or two corresponding edges - depending on whether the edge has *no points* in any Feasible Coverage Region, is *partially* in a Feasible Coverage Region, or is *entirely* in a Feasible Coverage Region. We illustrate this operation in figure 8.

Feasible Coverage Frame: For a given transmitter p, this is the set of points that lie on its Voronoi Frame and outside the union of interference disks of its neighbors.

Formally, the Feasible Coverage Frame for transmitter p is given as:

$$\perp_g(p) = \perp(p) \setminus \{x \ : \ x \in \bigcirc^i(q), \forall q \in \Gamma(p)\}$$

where $\bigcirc^i(q)$ denotes the interference range of the transmitter at q.

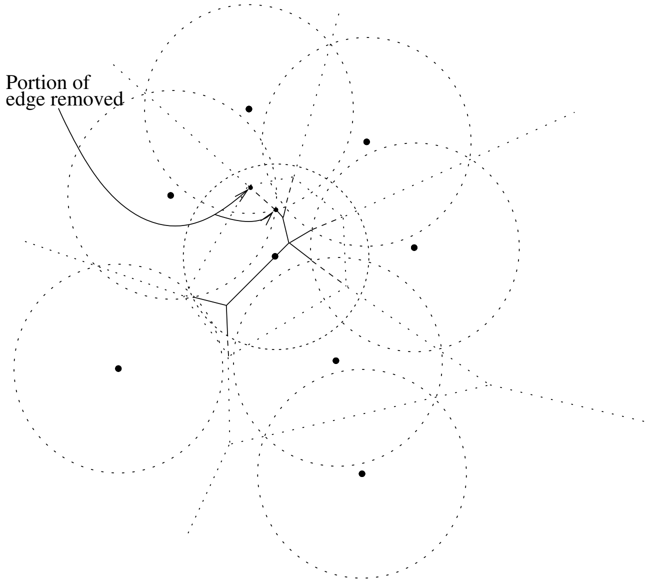

Figure 8: Voronoi Frame Yields Feasible Coverage Frame

We now have a procedure for building a Voronoi Frame for a transmitter, and for using this Voronoi Frame to find the transmitter's Feasible Coverage Frame. We note a property of the Voronoi Frame that we will use to show its correlation with the Feasible Coverage Region.

Observation 2.3. *$\perp(p)$ partitions $\triangle(p)$, and each partition corresponds to exactly one neighbor in $\Gamma(p)$.*

Proof. $\mathbb{V}(T \setminus \{p\})$, due to the Voronoi property, partitions the plane. By definition, the Voronoi Frame is the subset of this Voronoi diagram lying inside $\triangle(p)$. Hence $\triangle(p)$ is partitioned by the Voronoi Frame. Each point in a partition belongs to some Voronoi region in $\mathbb{V}(T \setminus \{p\})$. Due to lemma 2.2, a point in such a partition can be closest only to a neighbor of p in $\mathbb{V}(T)$.

Each point in a partition, can by definition, be closest only to one point in $T \setminus \{p\}$. Hence, the partition corresponds to exactly one neighbor. ☐

Note that the edges in the Voronoi Frame bounding this partition correspond to the edges contributed by a neighbor of p. Further, since p is closer than q to each point in this partition, the edge between p and q also bounds the partition.

This observation is illustrated by figure 9. We denote by $\triangle(p, q)$ the partition of $\triangle(p)$ by edges on the Voronoi Frame corresponding to q. Formally,

$$\triangle(p, q) = \triangle(p) \cap \{x \mid q \in \nu(x, \Gamma(p))\}$$

We note a correlation between $\triangle(p, q)$ and the interference disk for q.

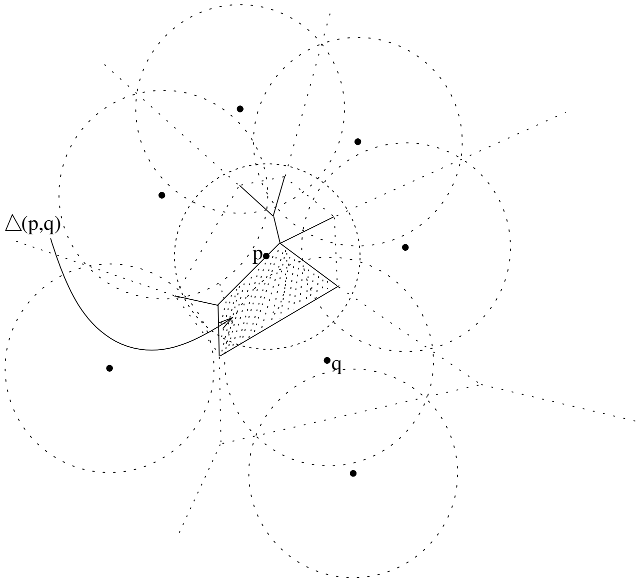

Figure 9: Voronoi Frame Partitions Voronoi Region

Corollary 2.2.2. *If $x \in \triangle(p,q)$ and x is in the interference range of some transmitter $t \neq p$, then x is in the interference range of q.*

Proof. By definition, every point in $\triangle(p,q)$ is closer to q than any other point in T. Since x is closer to q than t, we can apply observation 2.2 to see that x is in the interference range of q. $\Box$

We refer again to figures 4, 5, and 6. We observe that each Contiguous Feasible Region is bounded by the arcs of the rims of interference disks. These disks correspond to edges in the Feasible Coverage Frame enclosed within the Feasible Region.

We could compute the Actual Coverage Area of a transmitter directly by intersections of each Contiguous Feasible Region with the transmission disk. This would get us the coverage map. However, this would require our algorithm to represent the Contiguous Feasible Region as a sequence of arcs. Instead, we obtain the Actual Coverage Area by an alternative approach that uses the Voronoi properties.

We denote the Actual Coverage Area of a transmitter p by $\chi(p)$. The following result shows that the partition given by the Voronoi Frame allows us to compute the coverage area by excluding interference from just one transmitter.

Theorem 2.1.

$$\chi(p) = \bigcup_{q \in \Gamma(p)} \left(\bigcirc^t(p) \cap \triangle(p,q)\right) \setminus \bigcirc^i(q)$$

Proof. Let $\bigcirc^i(p)$ and $\bigcirc^t(p)$ denote the interference and transmission disks, respectively, of transmitter p. Lemma 2.1 implies that the Actual Coverage Area lies inside $\triangle(p)$. Also, observation 2.3 states that the Contiguous Feasible Region for p is composed of contributions from each neighbor. Corollary 2.2.2 shows that to find points within $\triangle(p,q)$ that lie in the Feasible Coverage Area, it is sufficient only to exclude points on the interference disk of q. Thus the Actual Coverage Area can be computed from the individual regions contributed by each partition. $\Box$

The advantage of using the Voronoi Frame is now clear - we need only the Feasible Frame to represent the Actual Coverage Area. Also, to compute the Actual Coverage Area, only two arc intersection computations are required for each neighbor q - one for $\bigcirc^t(p)$, and one for $\bigcirc^i(q)$.

In order to compute $\chi(p)$ we need a generalized polygon representation that allows circular arcs as edges. Berberich et al. ([5]) study intersections of polygons with arcs. We defer discussion on these generalized polygons until Section 4.

In this section we have shown how to compute the coverage map for a set of transmitters having the same interference (and transmission, respectively) radius. We generalize the arguments presented here to transmitters with unequal interference (and transmission, respectively) radii in Section 3.

3. UNEQUAL RANGES AND THE POWER DIAGRAM

Note that observation 2.1 does not hold when the interference radii (and transmission radii, respectively) are not the same. This is because equal minimum Euclidean distance of point x from two disks $\bigcirc(c_1, r_1)$ and $\bigcirc(c_2, r_2)$, where $r_1 \neq r_2$ does not imply equal Euclidean distance of the centers c_1 and c_2 from x. Thus, we need find an alternative distance measure.

The *power diagram* (see Aurenhammer et al. [3]) is a generalization of the closest point Voronoi diagram we have used in Section 2. The power diagram is based on a different distance measure (between a point and a disk), called the *power distance*. We denote the power distance by ρ. This section demonstrates the similarities between this distance measure (ρ) and the distance measure δ in Section 2, and the ensuing similarities between the power diagram and the Voronoi diagram. These similarities enable us to extend our ideas from Section 2 to unequal ranges.

The power distance of a point p from a disk $\bigcirc$ of radius r and center c is defined by $\rho(p, \bigcirc) = d(p,c)^2 - r^2$; where $d(p,c)$ is the Euclidean distance between p and the center c. Geometrically, the power distance of a point outside a disk is the square of the length of the tangent from that point to the disk rim. Inside the disk perimeter, the power distance is negative in sign, and corresponds to the square of half the length of the chord normal to the line joining the point and the center of the disk.

The Voronoi diagram for disks with equal radii under the distance measure δ is a power diagram. This is because the signs of $\rho(p, \bigcirc)$ and $\delta(p, \bigcirc)$ are the same, and $\rho(p, \bigcirc) - \rho(p, \bigcirc') = \delta(p, \bigcirc)^2 - \delta(p, \bigcirc')^2$ if the radii are equal.

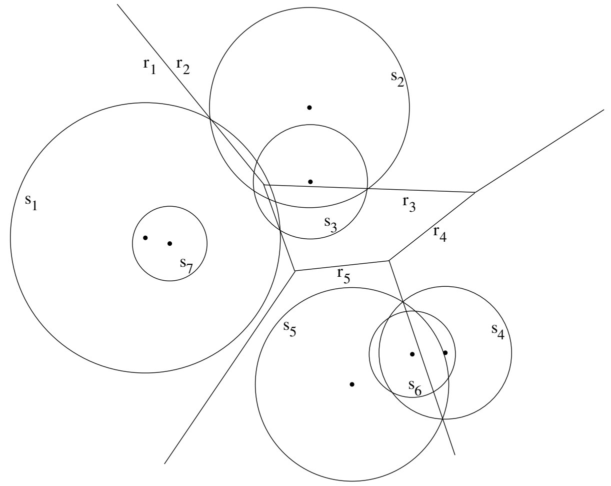

Figure 10: A Power Diagram: 7 disks, 5 regions (adapted from [3])

Figure 10 shows a power diagram of 7 disks in the plane. Some fundamental properties of the power diagram are stated in [3]. The power diagram for a set of disks, Ψ, partitions the plane into convex polygonal regions, i.e. shapes exactly like Voronoi regions. A point p lies in the *power region* corresponding to disk $\bigcirc \in \Psi$ if its power distance from $\bigcirc$ is less than its power distance from every other disk in Ψ.

The power diagram generalizes the Voronoi distance measure of δ to ρ. This leads us to three key facts, which together form the core of our progression from Voronoi diagrams to power diagrams as tools for computing the coverage map.

1. A generalization of observation 2.2 shows the same relationship exists between the Power Region and interference disks. This generalization is shown in observation 3.1.
2. A generalization of lemma 2.1 (lemma 3.1 also holds when the distance measure is replaced by ρ. This means that the Actual Coverage Area will lie only in the Power Region.
3. A generalization of lemma 2.2 is also possible, as we will see shortly in lemma 3.2. This means that we need to consider interference only from transmitters that are Power Neighbors.

In table 1, we introduce the new notation for unequal ranges and the power diagram. We also note the corresponding notation with equal ranges and the Voronoi diagram.

We begin with an observation that relates the distance measure ρ to interference disks. Note its correspondence with observation 2.2

Observation 3.1. *Consider a point x on the interference disk for $\tilde{p}$. x is also on the interference disk of every transmitter closer than $\tilde{p}$ in power distance.*

Equal Ranges	→	Unequal Ranges
Voronoi Diagram $\mathbb{V}(T)$	→	Power Diagram $\mathbb{P}(T)$
Transmitter Location p	→	Transmitter Location $\tilde{p}$
Voronoi region $\triangle(p)$	→	Power region $\triangle(\tilde{p})$
Voronoi neighbors $\Gamma(p)$	→	Power neighbors $\Gamma(\tilde{p})$
Voronoi Frame $\perp(p)$	→	Power Frame $\perp(\tilde{p})$
Voronoi region partition by Voronoi Frame $\triangle(p,q)$	→	Power region partition by Power Frame $\triangle(\tilde{p},\tilde{q})$
Actual Coverage Area	→	Actual Coverage Area
Feasible Coverage Area	→	Feasible Coverage Area
Contiguous Feasible Region	→	Contiguous Feasible Region

Table 1: Notation for Unequal Ranges

Proof. Let $\tilde{q} \in T \setminus \{\tilde{p}\}$. Let x be closer, by power distance measure ρ, to $\tilde{q}$ than $\tilde{p}$. Let $\bigcirc_{\tilde{p}}$ and $\bigcirc_{\tilde{q}}$ be the interference disks of $\tilde{p}$ and $\tilde{q}$, respectively.

Since $\rho(x, \bigcirc_{\tilde{q}}) < \rho(x, \bigcirc_{\tilde{p}})$, if x is in the interference range of $\tilde{p}$, $\rho(x, \tilde{p}) < 0$. Which means that $\rho(x, \tilde{q}) < 0$, i.e. x must also be in the interference range of $\tilde{q}$. ☐

We now observe the following generalization of lemma 2.1 -

Lemma 3.1 (Coverage in Power Region). *Consider a point x outside $\triangle(\tilde{p})$ that is on the transmission disk for $\tilde{p}$. x is also on some interference disk other than that of $\tilde{p}$.*

Proof. Let $\tilde{q} \in T \setminus \{\tilde{p}\}$, and $x \in \triangle(\tilde{q})$. Since the power diagram is a partition of the plane, such a $\tilde{q}$ exists. Thus, x is closer to $\tilde{q}$, in power distance, than it is to $\tilde{p}$. The transmission disk of $\tilde{p}$ is a subset of its interference disk; thus x lies on the interference disk of $\tilde{p}$. Hence, by observation 3.1 x must also be on the interference disk of $\tilde{q}$. ☐

In the ensuing discussion, we will implicitly assume the use of the distance measure ρ, the power distance; that is, we will say 'closest' or 'closer' (respectively, 'farthest' or 'farther') to mean closest or closer (farther, farthest, respectively) in power distance.

Lemma 3.2. *Assume that every transmitter has a non-empty power region. Let x be a point in the power region of transmitter $\tilde{p}$, i.e. $\triangle(\tilde{p})$. If $\tilde{q}$ is the closest (by power distance) transmitter in $T \setminus \{\tilde{p}\}$ to x, then $\tilde{q}$ is a power neighbor of $\tilde{p}$.*

Proof. We denote by $\triangle(\tilde{p}, T)$ the power region of $\tilde{p}$ in the power diagram of the set of transmitters T. Similarly, $\triangle(\tilde{q}, T \setminus \{\tilde{p}\})$ denotes the power region of $\tilde{q}$ in the set of transmitters $T \setminus \{\tilde{p}\}$. By definition, $x \in \triangle(\tilde{p}, T) \cap \triangle(\tilde{q}, T \setminus \{\tilde{p}\})$.

Assume that $\tilde{q}$ is not a power neighbor of $\tilde{p}$. We show that this leads to a contradiction.

Case 1 : Assume that an extreme point x_0 of $\triangle(\tilde{p}, T)$ lies in $\triangle(\tilde{q}, T \setminus \{\tilde{p}\})$. We show that this leads to a contradiction.

Let $\tilde{q_0}$ be a power neighbor of $\tilde{p}$ corresponding to the point x_0. Thus,

$$\rho(x_0, \tilde{p}) = \rho(x_0, \tilde{q_0})$$

However, since $\tilde{q}$ is closer to x_0 than $\tilde{q_0}$,

$$\rho(x_0, \tilde{p}) = \rho(x_0, q_0) > \rho(x_0, \tilde{q})$$

This is a contradiction since no transmitter can be closer to x_0 than $\tilde{p}$. Thus, no extreme points of $\triangle(\tilde{p}, T)$ lie in $\triangle(\tilde{q}, T \setminus \{\tilde{p}\})$.

Case 2 : Assume that no extreme point of $\triangle(\tilde{p}, T)$ lies in $\triangle(\tilde{q}, T \setminus \{\tilde{p}\})$. We show that this leads to a contradiction.

$\Rightarrow$ All edges in $\triangle(\tilde{p}, T)$ lie outside $\triangle(\tilde{q}, T \setminus \{\tilde{p}\})$.
$\Rightarrow$ Either $\triangle(\tilde{q}, T \setminus \{\tilde{p}\}) \subset \triangle(\tilde{p}, T)$, or $\triangle(\tilde{q}, T \setminus \{\tilde{p}\}) \cap \triangle(\tilde{q}, T \setminus \{\tilde{p}\}) = \phi$.
In other words, the power region of $\tilde{p}$ in $\mathbb{P}(T)$ either encloses that of $\tilde{q}$ in $\mathbb{P}(T \setminus \tilde{p})$, or the two power regions are disjoint.
Since $x \in \triangle(\tilde{p}, T) \cap \triangle(\tilde{q}, T \setminus \{\tilde{p}\}) \Rightarrow \triangle(\tilde{q}, T \setminus \{\tilde{p}\}) \cap \triangle(\tilde{q}, T \setminus \{\tilde{p}\}) \neq \phi$
Thus, $\triangle(\tilde{q}, T \setminus \{\tilde{p}\}) \subset \triangle(\tilde{p}, T)$
$\Rightarrow$ each point in $\triangle(\tilde{q}, T \setminus \{\tilde{p}\})$ is closer to $\tilde{p}$ than $\tilde{q}$.
$\Rightarrow \triangle(\tilde{q}, T) = \phi$
This contradicts the assumption that no power region is empty.

$\square$

Having established the basic correspondence between Voronoi diagrams and power diagrams, we note that observation 2.3, 2.2.2, and theorem 2.1 are directly applicable to power diagrams. We give corresponding results next. The proofs for these results are almost identical to earlier proofs.

OBSERVATION 3.2. *$\perp(\tilde{p})$ partitions $\triangle(\tilde{p})$, and each partition corresponds to exactly one neighbor in $\Gamma(\tilde{p})$.*

PROOF. $\mathbb{P}(T \setminus \{\tilde{p}\})$, by definition, partitions the plane. By definition, the Power Frame is the subset of this power diagram lying inside $\triangle(\tilde{p})$. Hence $\triangle(\tilde{p})$ is partitioned by the Power Frame. Each point in a partition belongs to some power region in $\mathbb{P}(T \setminus \{\tilde{p}\})$. Due to lemma 3.2, a point in such a partition can be closest only to a neighbor of p in $\mathbb{P}(T)$.

Each point in a partition, can by definition, be closest only to one point in $T \setminus \{\tilde{p}\}$. Hence, the partition corresponds to exactly one neighbor. $\square$

We note that the same relationship as corollary 2.2.2 holds between $\triangle(\tilde{p}, \tilde{q})$ and the interference disk for $\tilde{q}$.

COROLLARY 3.2.1. *If $x \in \triangle(\tilde{p}, \tilde{q})$ and x is in the interference range of some transmitter $\tilde{t} \neq \tilde{p}$, then x is in the interference range of $\tilde{q}$.*

PROOF. By definition, every point in $\triangle(\tilde{p}, \tilde{q})$ is closer to $\tilde{q}$ than any other point in T. Since x is closer to $\tilde{q}$ than $\tilde{t}$, we can apply observation 3.1 to see that x is in the interference range of $\tilde{q}$. $\square$

We now state and prove our main theorem. Note that it is a generalization of theorem 2.1.

We denote the Actual Coverage Area of a transmitter $\tilde{p}$ by $\chi(\tilde{p})$. The following result shows that, if no power region is empty, then the partition given by the Power Frame allows us to compute the coverage area by excluding interference from just one transmitter.

THEOREM 3.1. *Assume no power region is empty. Then,*

$$\chi(\tilde{p}) = \bigcup_{q \in \Gamma(\tilde{p})} (\bigcirc^t(\tilde{p}) \cap \triangle(\tilde{p}, \tilde{q})) \setminus \bigcirc^i(\tilde{q})$$

PROOF. Let $\bigcirc^i(\tilde{p})$ and $\bigcirc^t(\tilde{p})$ denote the interference and transmission disks, respectively, of transmitter $\tilde{p}$. Since no power region is empty, lemma 3.1 implies that the Actual Coverage Area lies inside $\triangle(\tilde{p})$. Also, observation 3.2 states that the Contiguous Feasible Region for $\tilde{p}$ is composed of contributions from each neighbor. Corollary 3.2.1 shows that to find points within $\triangle(\tilde{p}, \tilde{q})$ that lie in the Feasible Coverage Area, it is sufficient only to exclude points on the interference disk of $\tilde{q}$. Thus the Actual Coverage Area can be computed from the individual regions contributed by each partition. $\square$

3.1 Removing Redundant Transmitters

The power region corresponding to a circle may be empty - as is the case with s_7 in figure 10. No point on the power bisector of s_1 and s_7 appears in the power diagram since each point on this bisector is closer to either s_2, s_3, or s_5.

However, as we will show below, if a disk $\bigcirc$ has an empty power region then it is included in the union of other disks in Ψ. A disk that belongs in the union of other disks has an empty coverage map, since every point on it is in the interference range of some other transmitter. Hence, for our purposes, during preprocessing we can remove disks that do not have a corresponding power region.

We show a more general result - that if a disk and its corresponding power region have no points in common, then that disk is included in the union of other disks.

LEMMA 3.3 (EMPTY POWER REGIONS). *Let $\tilde{q}$ be a transmitter with interference disk $\bigcirc_{\tilde{q}}$, such that $\bigcirc_{\tilde{q}} \cap \triangle(\tilde{q}) = \phi$. Then,*

$$\bigcirc_{\tilde{q}} \subseteq \bigcup_{\tilde{p} \in T \setminus \tilde{q}} \bigcirc_{\tilde{p}}$$

PROOF. We prove this result by contradiction.

Let $\bigcirc_{\tilde{q}} \nsubseteq \bigcup_{\tilde{p} \in T \setminus \tilde{q}} \bigcirc_{\tilde{p}}$
$\Rightarrow \exists x \in \bigcirc_{\tilde{q}}$ such that $\forall \tilde{p} \neq \tilde{q},\ x \notin \bigcirc_{\tilde{p}}$

$\Rightarrow (\rho(x, \bigcirc_{\tilde{q}}) < 0) \wedge (\forall \tilde{p} \neq \tilde{q},\ \rho(x, \bigcirc_{\tilde{q}}) > 0)$
$\Rightarrow x \in \triangle(\tilde{q})$
$\Rightarrow x \in \triangle(\tilde{q}) \cap \bigcirc_{\tilde{q}}$
This contradicts the assumption that $\triangle(\tilde{q}) \cap \bigcirc_{\tilde{q}}$ is empty. □

This fact justifies our pre-processing step for removing disks that have an empty power region.

4. ALGORITHM

We collate the observations made in the preceding text into an algorithm. The inputs to the algorithm are: a set T of transmitters, their locations in the plane, and their transmission and interference radii. The algorithm outputs a coverage map for T, denoted by $\widehat{\chi}(T)$.

The notation in table 1 is carried over as it is in the algorithm.

ALGORITHM 4.1 (COVERAGE MAP).

```
1. Initialize: $\widehat{\chi}(T) \leftarrow \phi$.
2. Compute the power diagram $\mathbb{P}(T)$.
3. For each transmitter $\tilde{p} \in T$, do If $\triangle(\tilde{p}) = \phi$, $T \leftarrow T \setminus \{\tilde{p}\}$.
4. For each transmitter $\tilde{p} \in T$, do
    (a) $\chi(\tilde{p}) \leftarrow \phi$
    (b) Find the power diagram of $\Gamma(\tilde{p})$, i.e.$\mathbb{P}(\Gamma(\tilde{p}))$.
    (c) For each region $\triangle(\tilde{q}, \Gamma(\tilde{p}))$, do
        i. $\triangle(\tilde{p}, \tilde{q}) \leftarrow \triangle(\tilde{q}, \Gamma(\tilde{p})) \cap \triangle(\tilde{p}, T)$
        ii. $\chi(\tilde{p}) \leftarrow \chi(\tilde{p}) \cup (\triangle(\tilde{p}, \tilde{q}) \cap \bigcirc^{t}_{\tilde{p}}) \setminus \bigcirc^{i}_{\tilde{q}})$
5. For each transmitter $\tilde{p} \in T$, do $\widehat{\chi}(T) \leftarrow \widehat{\chi}(T) \cup \chi(\tilde{p})$
```

4.1 Running Time Analysis

We show a result from [3] that bounds the number of power edges in a power diagram. This observation is part of lemma 1 in [3], and a detailed discussion can be found there. We just give a proof sketch here.

OBSERVATION 4.1. *The number of power edges in a power diagram is less than* $3n - 6$.

PROOF. The power diagram in the plane is a planar graph. Its dual graph $D(\mathbb{P})$ contains exactly one vertex for each region of $\mathbb{P}$. Two vertices of $D(\mathbb{P})$ are connected by an edge if, and only if, the boundaries of the corresponding regions of $\mathbb{P}$ have an edge in common. $D(\mathbb{P})$ is a triangulation on n vertices. A triangulation on n vertices cannot have more than $3n - 6$ edges. □

We make an observation that the sum of the number of neighbors over all transmitters is linear in n. This result will be invoked in our proof.

OBSERVATION 4.2 (SUM OF NEIGHBORS).

$$\sum_{\tilde{p} \in T} |\Gamma(\tilde{p})| = O(n)$$

PROOF. The sum $\sum\limits_{\tilde{p} \in T} |\Gamma(\tilde{p})|$ is also the number of ordered pairs $(\tilde{p}, \tilde{q})$ such that $\tilde{p}$ and $\tilde{q}$ are neighbors in $\mathbb{P}(T)$. Since each power edge in $\mathbb{P}(T)$ corresponds to two transmitters, the latter is twice the number of power edges, which is $O(n)$ by observation 4.1 □

THEOREM 4.1 (RUNTIME). *The coverage map of 'n' transmitters with equal or unequal ranges can be constructed in* $O(n \log n)$ *time.*

PROOF. Step 2 A power diagram of n disks in the plane can be constructed in $O(n \log n)$ time ([7]).

Step 4b The power diagram of $\Gamma(\tilde{p})$ can be constructed in $O(|\Gamma(\tilde{p})| \log |\Gamma(\tilde{p})|$ time.

Step 4(c)i A well-known algorithm ([6]) for convex polygon intersection can be used to compute the partition. This algorithm is linear in the total number of vertices, i.e. in our case $O(|\Gamma(\tilde{p})|)$.

Step 4(c)ii The union and set difference operations in 4(c)ii can be performed by the sweep-line algorithm from [5]. This computation is also linear time in the number of line segments (edges) and arcs; i.e. in our case $O(|\Gamma(\tilde{p})|)$.

Step 4 The loop 4 is executed once for each transmitter $\tilde{p}$. We show that total time taken to compute the Actual Coverage Areas of all transmitters is $O(n \log n)$.

We observe that the total number of edges and partitions added for each transmitter are most $O(n)$. Each transmitter $\tilde{p}$ can contribute a partition only to a neighbor (lemma 3.1). Thus the total number of partitions created by the algorithm is the sum of neighbors, which $O(n)$ by observation 4.2. Since each edge appears in at most two partitions, the total number of edges created is also $O(n)$.

The total time to compute the power diagrams for all transmitters is $O(\sum\limits_{\tilde{p} \in T} |\Gamma(\tilde{p})| \log |\Gamma(\tilde{p})|)$.
Now -
$\log |\Gamma(\tilde{p})| \leq \log n$
$\Rightarrow \sum\limits_{\tilde{p} \in T} |\Gamma(\tilde{p})| \log |\Gamma(\tilde{p})| \leq (\log n) \sum\limits_{\tilde{p} \in T} |\Gamma(\tilde{p})|$
$\sum\limits_{\tilde{p} \in T} |\Gamma(\tilde{p})| = O(n)$, by observation 4.2
$\Rightarrow (\log n) \sum_{\tilde{p} \in T} |\Gamma(\tilde{p})| = O(n \log n)$

Since the total time required to compute the Actual Coverage Areas of all transmitters is $O(n \log n)$, and all other operations require a total time of $O(n)$, the computation of the coverage map can be achieved in $O(n \log n)$ time. □

5. RELATED WORK AND FUTURE DIRECTIONS

Voronoi and power diagrams have been applied by So et al. in [1] to solve a sensor network coverage problem. Their model does not include interference constraints. They determine whether a given set of n sensors in a field $\mathbb{A}$ covers all points in $\mathbb{A}$ such that each point is covered by *at least k* sensors. However, even for $k = 1$, computing interference-limited coverage requires us to ensure that *exactly one disk* includes a point, which is a slightly harder problem.

Das et al. ([12]) consider the problem of power assignment to base stations. Their model does not include interference. Ahmed et al. ([4]) study algorithms for transmission power assignment to access points (APs) under interference constraints. They select a random AP to increase power if more coverage is required, and decrease power *by an arbitrary amount* if it interferes with an existing power assignment. The model chosen there too is a protocol model. Our results can directly be applied to this problem - exactly computing how much power needs to be increased or decreased for an AP without the need for choosing arbitrary powers iteratively. They also compute a utility function that needs a coverage map. They use sampling in the abscence of a known algorithm for computing the coverage map.

Coverage in a wireless network under interference constraints has also been studied in the combinatorial optimization model by Kuhn et al. ([8]). Their goal is to solve a minimax optimization problem - to obtain a power assignment for each base station such that the maximum number of base stations interfering with any receiver is minimized. The model is discrete and posed as a combinatorial choice of an appropriate subset of finite sets representing coverage areas.

5.1 Future Work

1. An immediate extension to this work is to allow upto k transmitters to interfere with a receiver. Higher-order Voronoi and power diagrams seem to be the natural tools to wield, as indicated by the generalizations in [1].

2. Our (protocol) model assumes that transmission and interference ranges of every transmitter are disks centered at the transmitter location, with signal strength constant at every point within a disk. Generalizations of the Voronoi diagram of point sets to that of a set of convex objects is studied in the literature (see [11]). We also seek to model the decrease of signal strength with distance and time in a way that will allow us to reason with geometric predicates, as in the case of Voronoi and power diagrams. These structures would take us a step closer to modeling the physical layer - both in the contour of the coverage regions and received signal strength.

3. Under a similar protocol model, [9] gives an assignment of transmitter locations, receiver locations, transmission radii, interference parameters, and transmission schedule that maximize the *transport* capacity of the arrangement to achieve an asymptotic upper bound on maximum transport capacity achievable by n wireless nodes under *any* arrangement. It will be interesting to find receiver locations and a transmission schedule that maximizes the transport capacity for any given assignment to the other three parameters.

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